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Generalized prequantization on principal bundles over homogeneous spaces and its applications *

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In this paper, we set up a framework of generalized prequantization by starting from a single structure on the principal bundle associated with the Lie group pair (G, H). This procedure could be regarded as a kind of "postulated rules" taking the place of ordinary prequantization. Illustrative examples which show the usefulness of our scheme are, among others, compact semisimple Lie groups and loop groups. The model spaces of loop groups are also discussed.

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Introduction

Geometric quantization in its well established form starts from a symplectic manifold (M, ω), and a complex line bundle over M (cf. refs. [13, 17]). In case of a co-adjoint orbit $M = G/H$, due to Kirillov's coorbit theory [6], what one gets after quantization is a certain representation of G, which is usually only one sector of the quantum mechanical Hilbert space. And it is always non-trivial to find a global form of the connection which meets the need of prequantization [13, 17].

In this paper, we set up a framework of generalized prequantization by starting from a single structure on the principal bundle P associated with the Lie group pair (G, H), which takes advantage of the Ehresmann connection induced from the G-actions on P. This procedure could be regarded as a kind of "postulated rules" taking the place of ordinary prequantization [13, 17].

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The conditions under which our suggested procedure can really be carried out are given in this paper. Illustrating examples which show the usefulness of our scheme are, among others,

1. a short and transparent treatment of geometric quantization and the Borel–Weil theorem for compact groups (other finite dimensional cases could easily be found also satisfying our conditions) (cf. refs. [1, 3, 12])
2. the Borel–Weil theorem for loop groups LG ($M = LG/T$) as treated in ref. [10],
3. geometric quantization of the Wess–Zumino–Witten (WZW) model, whose correct phase space should be LG/G instead of LG/T as most people seem to take for granted [4, 9–11, 15, 16], and
4. the model spaces of loop groups and the model space of the WZW model [11].

Actually we believe that all kinds of induced representations are susceptible to such explanations.

1. Generalized prequantization on principal bundles over homogeneous spaces

Let G be a Lie group, and H be a closed Lie subgroup of G. Then G can be identified with a principal bundle on the homogeneous space G/H of G (the left coset space of H) taking H as both its fiber and its structure group. The projection mapping $\pi : G \longrightarrow G/H$ is just the natural quotient mapping induced by H. The left actions of G on G induce the actions of G on G/H, and they commute with π.

The right actions of H on G are just the actions of the structure group H on the principal bundle $\pi : G \longrightarrow G/H$. Thus at each point g of G, we identify the Lie algebra $\mathcal{H}$ of H with the subspace of the tangent space $T_g G$ as follows:

$$\sigma_g : \mathcal{H} \longrightarrow T_g G, \quad \xi \mapsto \sigma_g(\xi), \tag{1.1}$$

where $\sigma_g(\xi)$ is the tangent vector of $T_g G$ corresponding to the one-parameter current $g \exp(-t\xi)$ of G for $t \in R^1$ [1, 14].

Let $\mathcal{G}$, $\mathcal{H}$ be the Lie algebras of the Lie groups G, H respectively. Suppose that there exists a direct sum decomposition of $\mathcal{G}$ which satisfies

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{M}, \qquad [\mathcal{H}, \mathcal{M}] \subseteq \mathcal{M}. \tag{1.2}$$

We define a $\mathcal{H}$-valued one-form α on G as

$$\alpha(L_\xi)(g) \stackrel{\text{def}}{=} -\xi_{\mathcal{H}} \quad \text{for all } \xi \in \mathcal{G}, \text{ all } g \in G, \tag{1.3}$$

where L_ξ is the left-invariant vector field on G with $L_\xi(e) = \xi$, and $\xi_{\mathcal{H}}$ is the $\mathcal{H}$-component of ξ under the decomposition of $\mathcal{G}$ in (1.2). Then α is an Ehresmann connection in the principal bundle $\pi : G \longrightarrow G/H$ [14], such that

$$\begin{aligned} \alpha(\sigma(\xi)) &= \xi, \\ \alpha(R_{h*}X)(g) &= \mathrm{Ad}\, h.^{-1}\alpha(X)(g), \end{aligned} \tag{1.4}$$

for all $\xi \in \mathcal{H}$, all $h \in H$, all $g \in G$, and all tangent vectors $X \in T_gG$, where the mapping σ is defined in (1.1), R_h denotes the right action of h on G, and R_{h*} is the tangent mapping of R_h. Due to ref. [14], the connection α determines a horizontal space decomposition of T_gG for each $g \in G$ as follows:

$$T_gG = H_g \oplus V_g, \tag{1.5}$$

where the vertical space $V_g \overset{\text{def}}{=} \{\sigma_g(\xi) \mid \xi \in \mathcal{H}\}$, and the horizontal space $H_g \overset{\text{def}}{=} \ker \alpha_g$.

According to ref. [14], the curvature of the connection α is defined as a $\mathcal{H}$-valued two-form Ω on G such that

$$\Omega(X,Y)(g) \overset{\text{def}}{=} \mathrm{d}\alpha(\pi_{H_g}(X), \pi_{H_g}(Y))(g) \tag{1.6}$$

for all $g \in G$, and all tangent vectors $X,\ Y \in T_gG$, where Π_{H_g}: $T_gG \rightarrow H_g$ is the projection. Then we have

$$\begin{aligned} \Omega(L_\xi,\ L_\eta) &= [\xi_{\mathcal{M}},\ \eta_{\mathcal{M}}]_{\mathcal{H}}, \\ \Omega(R_{h*}X,\ R_{h*}Y)(g) &= \mathrm{Ad}\, h.^{-1}\Omega(X,\ Y)(g), \end{aligned} \tag{1.7}$$

for all $\xi,\ \eta \in \mathcal{G}$, all $h \in H$, all $g \in G$, and all tangent vectors $X,\ Y \in T_gG$, where $\xi_{\mathcal{M}}$ denotes the $\mathcal{M}$-component of ξ.

Generally speaking, the curvature Ω on the principal bundle $\pi : G \longrightarrow G/H$ is not the pullback of a $\mathcal{H}$-valued two-form ω on G/H by the mapping π. However, in the case discussed below, the curvature Ω is really the pullback of a $\mathcal{H}$-valued two-form ω on G/H, which could be regarded as the basic structure of generalized prequantization of the principal bundle $\pi : G \longrightarrow G/H$ (the counterpart of ordinary symplectic structures of classical phase spaces).

By the left actions of G on G, and the induced actions of G on G/H, we define the infinitesimal actions of $\mathcal{G}$ on $C^\infty(G)$ and $C^\infty(G/H)$, respectively, as

$$\begin{aligned} \xi_G(F)(g) &\overset{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}t}F(\exp(-t\xi)\, g)|_{t=0}, \\ \xi_{G/H}(f)(\pi(g)) &\overset{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}t}f \circ \pi(\exp(-t\xi)\, g)|_{t=0}, \end{aligned} \tag{1.8}$$

for all $\xi \in \mathcal{G}$, all $g \in G$, all $F \in C^\infty(G)$, and all $f \in C^\infty(G/H)$.

In fact, we can relate $\xi_{G/H}$ to ξ_G with the connection α and details are summarized in the following proposition of generalized Kostant mappings of the principal bundle $\pi : G \longrightarrow G/H$.

Proposition 1.1 (Generalized Kostant mappings). *Suppose that the Lie algebras $\mathcal{G}, \mathcal{H}$ of the Lie groups G, H satisfy (1.2), and that*

$$[\mathcal{M}, \mathcal{M}] \subseteq \mathcal{M} \oplus \mathcal{Z}, \tag{1.9}$$

$$\mathcal{H} = \mathcal{Z} \oplus \mathcal{Y}, \qquad [\mathcal{H}, \mathcal{Y}] \subseteq \mathcal{Y}, \tag{1.10}$$

where $\mathcal{Z}$ is the center of $\mathcal{H}$.

We define two linear mappings $\delta_G : \mathcal{G} \longrightarrow C^\infty(G, \mathcal{H})$ and $\delta_{G/H} : \mathcal{G} \longrightarrow C^\infty(G/H, \mathcal{Z})$, respectively, as

$$\begin{aligned} \delta_G(\xi)(g) &\stackrel{\text{def}}{=} -(\operatorname{Ad} g.^{-1}\xi)_{\mathcal{H}}, \\ \delta_{G/H}(\xi)(\pi(g)) &\stackrel{\text{def}}{=} -(\operatorname{Ad} g.^{-1}\xi)_{\mathcal{Z}}, \end{aligned} \tag{1.11}$$

for all $\xi \in \mathcal{G}$, and all $g \in G$. Then:

(1) The curvature Ω of the connection α defined in (1.6) is the pullback of a $\mathcal{Z}$-valued two-form ω on G/H, which satisfies

$$\omega(\pi_*(L_\xi(g)), \pi_*(L_\eta(g)))(\pi(g)) = [\xi_{\mathcal{M}}, \eta_{\mathcal{M}}]_{\mathcal{Z}} \tag{1.12}$$

for all ξ, $\eta \in \mathcal{G}$, and all $g \in G$.

(2) For all $\xi \in \mathcal{G}$, and all $g \in G$,

$$\begin{aligned} \xi_G(g) &= \tilde{\xi}_{G/H}(g) + L_{\delta_G(\xi)}(g) \\ &= \tilde{\xi}_{G/H}(g) - \sigma(\delta_G(\xi))(g), \end{aligned} \tag{1.13}$$

$$\mathrm{d}\delta_{G/H}(\xi) = \mathrm{i}(\xi_{G/H})\omega, \tag{1.14}$$

where $\tilde{\xi}_{G/H}$ denotes the horizontal lift of the vector field $\xi_{G/H}$ on G/H determined by the horizontal decomposition of T_gG in (1.5) corresponding to the Ehresmann connection α. Due to the identity (1.14) it makes sense to call the vector field $\xi_{G/H}$ on G/H a generalized Hamiltonian vector field of $\delta_{G/H}(\xi)$ corresponding to ω. (cf. refs. [13, 17]). Due to the identities (1.13) and (1.14), we call the mappings δ_G, and $\delta_{G/H}$ generalized Kostant mappings of the principal bundle $\pi : G \longrightarrow G/H$ (cf. refs. [6, 13, 17]).

The proof is by direct calculation, we omit it.

Corollary 1.2. *Suppose that the Lie algebras $\mathcal{G}$, $\mathcal{H}$ satisfy the conditions in (1.2), and H is abelian. Then the mappings δ_G, $\delta_{G/H}$ defined in (1.12) satisfy*

$$\delta_G(\xi) = \delta_{G/H}(\xi) \circ \pi \quad \text{for all } \xi \in \mathcal{G}, \tag{1.15}$$

$$\begin{aligned} \xi_G(g) &= \tilde{\xi}_{G/H}(g) + L_{\delta_{G/H}(\xi)\circ\pi(g)}(g) \\ &= \tilde{\xi}_{G/H}(g) - \sigma(\delta_{G/H}(\xi)(\pi(g))(g). \end{aligned} \tag{1.16}$$

Remark. If the Lie algebras $\mathcal{G}$, $\mathcal{H}$ of the Lie groups G, H satisfy the conditions supposed in proposition 1.1, then we can associate the left infinitesimal actions of $\mathcal{G}$ on $C^\infty(G)$ with their induced actions of $\mathcal{G}$ on $C^\infty(G/H)$ through the Ehresmann connection α in proposition 1.1. This procedure is similar to ordinary prequantization of a complex line bundle on a symplectic manifold [13, 17]. However, since the $\mathcal{Z}$-valued two-form ω on G/H (corresponding to the curvature Ω of the connection α) in (1.12) may be degenerate, and the dimension of the fiber H of the principal bundle $\pi : G \longrightarrow G/H$ may be larger than 1, we would like to call this procedure generalized prequantization of the principal bundle G on its homogeneous space G/H.

Definition 1.3. (Poisson structure of a subspace of $C^\infty(G/H, \mathcal{Z})$, and generalized prequantization operators) Assume that the Lie algebras $\mathcal{G}$, $\mathcal{H}$ satisfy the conditions (1.2), (1.9) and (1.10). We define a subspace $C^\infty_{\mathcal{G}}(G/H, \mathcal{Z})$ of $C^\infty(G/H, \mathcal{Z})$ as

$$C^\infty_{\mathcal{G}}(G/H, \mathcal{Z}) \stackrel{\text{def}}{=} \{\delta_{G/H}(\xi) \mid \xi \in \mathcal{G}\}. \tag{1.17}$$

Then

(1) For all $\xi, \eta \in \mathcal{G}$, we define the Poisson bracket of two $\mathcal{Z}$-valued smooth functions $\delta_{G/H}(\xi)$ and $\delta_{G/H}(\eta)$ as

$$\{\delta_{G/H}(\xi),\ \delta_{G/H}(\eta)\} \stackrel{\text{def}}{=} \omega(\eta_{G/H},\ \xi_{G/H}), \tag{1.18}$$

where ω is a $\mathcal{Z}$-valued two-form on G/H defined in (1.12), and $\xi_{G/H}$ defined in (1.8) is the generalized Hamiltonian vector field of $\delta_{G/H}(\xi)$ which satisfies (1.14).

(2) For all prequantization operators $\hat{\delta}_{G/H}(\xi)$ on $C^\infty(G)$ as

$$\hat{\delta}_{G/H}(\xi) \stackrel{\text{def}}{=} \tilde{\xi}_{G/H} + L_{\delta_G(\xi)} = \xi_G, \tag{1.19}$$

where ξ_G is defined in (1.8), $\delta_G(\xi)$ in (1.11), and $\tilde{\xi}_{G/H}$ is the horizontal lift of $\xi_{G/H}$.

Theorem 1.4. *Suppose that the Lie algebras $\mathcal{G}, \mathcal{H}$ of the Lie groups G, H satisfy the conditions (1.2), (1.9) and (1.10). Then:*

(1) for all $\xi, \eta \in \mathcal{G}$,

$$\{\delta_{G/H}(\xi),\ \delta_{G/H}(\eta)\} = \delta_{G/H}([\xi,\ \eta]); \tag{1.20}$$

moreover, the Lie algebra $(C^\infty_{\mathcal{G}}(G/H, \mathcal{Z}), \{\ ,\ \})$ is Lie-isomorphic to $\mathcal{G}$;

(2) the generalized prequantization mapping $\mathcal{Q} : \delta_{G/H}(\xi) \mapsto \hat{\delta}_{G/H}(\xi)$ is a Lie homomorphism from $(C^\infty_{\mathcal{G}}(G/H, \mathcal{Z}), \{\ ,\ \})$ to the space of linear differential operators on $C^\infty(G)$.

Proof.

(1) By (1.10), it is implied that

$$[\mathcal{H}, \mathcal{H}] \subseteq \mathcal{Y}. \tag{1.21}$$

Then it follows from (1.2) and (1.21) that the identity

$$[\xi, \eta]_{\mathcal{Z}} = [\xi_{\mathcal{M}}, \eta_{\mathcal{M}}]_{\mathcal{Z}} \tag{1.22}$$

holds for all $\xi, \eta \in \mathcal{G}$. Since

$$\begin{aligned}
&\{\delta_{G/H}(\xi), \delta_{G/H}(\eta)\}(\pi(g)) \\
&\quad = \omega(\eta_{G/H}, \xi_{G/H})(\pi(g)) \\
&\quad = \omega(-\pi_*(L_{(\mathrm{Ad}\, g.^{-1}\eta)_{\mathcal{M}}}(g)), -\pi_*(L_{(\mathrm{Ad}\, g.^{-1}\xi)_{\mathcal{M}}}(g))) \\
&\quad = [(\mathrm{Ad}\, g.^{-1}\eta)_{\mathcal{M}}, (\mathrm{Ad}\, g.^{-1}\xi)_{\mathcal{M}}]_{\mathcal{Z}} \\
&\quad = [\mathrm{Ad}\, g.^{-1}\eta, \mathrm{Ad}\, g.^{-1}\xi]_{\mathcal{Z}} \\
&\quad = (-\mathrm{Ad}\, g.^{-1}[\xi, \eta])_{\mathcal{Z}} \\
&\quad = \delta_{G/H}([\xi, \eta]),
\end{aligned}$$

it follows that the identity (1.20) holds for all $\xi, \eta \in \mathcal{G}$. Moreover, $(C^\infty_{\mathcal{G}}(G/H, \mathcal{Z}), \{\ ,\ \})$ is a Lie algebra, and the mapping $\psi : \xi \mapsto \delta_{G/H}(\xi)$ is a Lie isomorphism from $\mathcal{G}$ to $C^\infty_{\mathcal{G}}(G/H, \mathcal{Z})$.

(2) Since ξ_G is the representation of $\mathcal{G}$ on $C^\infty(G)$, it follows from (1.19) and (1.20) that the mapping $\mathcal{Q}$ is a Lie homomorphism. □

Remark. Our generalized prequantization is just the representation of $\mathcal{G}$ on $C^\infty(G)$ by (1.19). Definition 1.3 and theorem 1.4 show that our generalized prequantization scheme is similar to the ordinary prequantization scheme [13, 17] in the definitions and properties of the Poisson bracket and prequantization operators. Since the $\mathcal{Z}$-valued two-form ω on G/H may be degenerate, the Poisson bracket and generalized prequantization operators in our scheme are only defined in a subspace $C^\infty_{\mathcal{G}}(G/H, \mathcal{Z})$ of $C^\infty(G/H, \mathcal{Z})$, which is not an algebra. An interesting question is how to extend the definitions of the Poisson bracket and generalized prequantization operators to a linear space which is larger than $C^\infty_{\mathcal{G}}(G/H, \mathcal{Z})$, is a Poisson algebra, and guarantees a result similar to theorem 1.4(2).

The following section is motivated by the question whether there exist some illustrative examples of the Lie group pairs (G, H), whose Lie algebras $\mathcal{G}$, $\mathcal{H}$ satisfy the conditions (1.2), (1.9) and (1.10) supposed in proposition 1.1. The answer to this question is actually positive. In fact, the dimensions of the interesting examples range from finite dimensions to infinite dimensions. In particular, applying this method to loop groups [9, 10], we make contributions to the geometric quantization of the Wess–Zumino–Witten model in two-dimensional conformal field theory (cf. refs. [11, 16]).

2. Applications of generalized prequantization

2.1. COMPACT SEMISIMPLE LIE GROUPS

Let G be a compact semisimple Lie group, and T be the maximal torus of G. It is easy to see that the principal bundle $\pi : G \longrightarrow G/H$ satisfies the conditions supposed in theorem 1.2. In fact, due to Adams [1], the Lie algebras $\mathcal{G}$, $\mathcal{T}$ of G, T satisfy

$$\begin{aligned} \mathcal{G} &= \mathcal{T} \oplus \mathcal{M}, \\ [\mathcal{T}, \mathcal{T}] &= \{0\}, \\ [\mathcal{T}, \mathcal{M}] &= \mathcal{M}, \\ [\mathcal{M}, \mathcal{M}] &\subseteq \mathcal{M} \oplus \mathcal{T}, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} \mathcal{M} \stackrel{\text{def}}{=} \operatorname{span}\{ &(e_\beta - e_{-\beta}),\ \mathrm{i}(e_\beta + e_{-\beta}) \mid \beta \text{ is a positive root of } \mathcal{G}_{\mathbb{C}}, \\ &\text{and } e_\beta,\ e_{-\beta} \text{ are the root vectors of } \mathcal{G}_{\mathbb{C}} \\ &\text{corresponding to the roots } \beta \text{ and } -\beta\}. \end{aligned}$$

We see that the connection (1.2), (1.9) and (1.10) are satisfied, thus the connection α and generalized Kostant mappings of this principal bundle can be obtained through proposition 1.1. Moreover, since (2.1) make the conditions in corollary 1.3 satisfied, generalized Kostant mappings are only determined by the linear mapping $\delta_{G/T} : \mathcal{G} \longrightarrow C^\infty(G/T, \mathcal{T})$, i.e.,

$$\delta_{G/T}(\xi)(g) = -(\mathrm{Ad}\, g.^{-1}\xi)_{\mathcal{T}}$$

for all $g \in G$, and all $\xi \in \mathcal{G}$.

If λ is a weight of G, then we obtain the induced complex line bundle $\mathbb{C} \longrightarrow L_\lambda \twoheadrightarrow G/T$ equipped with the induced connection $\nabla^{(\lambda)}$ and induced Kostant mapping. It implies that the line bundle L_λ is a "generalized" prequantization line bundle on G/T, where the word "generalized" means that the curvature ω_λ of the induced connection may not be a symplectic form on G/T.

We choose the Kählerian polarization P on G/T [13, 17] as

$$P(\pi(g)) \stackrel{\text{def}}{=} \{\pi_*(L_\xi(g)) \mid \xi \in \mathcal{N}_+\}, \tag{2.2}$$

where $\mathcal{N}_+$ is the positive root vector subspace of $\mathcal{G}_{\mathbb{C}}$. The polarization space

$$\begin{aligned} \Gamma_0^P(L_\lambda) \stackrel{\text{def}}{=} \{&F \in \Gamma(L_\lambda) \mid \nabla^{(\lambda)}_{X(\pi(g))} F(g) = 0 \\ &\text{for all tangent vectors } X(\pi(g)) \in P(\pi(g))\}, \end{aligned} \tag{2.3}$$

where $\Gamma(L_\lambda)$ is the smooth section space of the complex line bundle L_λ, which can be identified with the following subspace of the smooth function space of G:

$$\Gamma(L_\lambda) \cong \{F \in C^\infty(G) \mid F(gt) = \lambda^{-1}(t)F(g) \text{ for all } t \in T, \text{ all } g \in G\}. \quad (2.4)$$

In fact, the polarization space $\Gamma_0^P(L_\lambda)$ is just the holomorphic section space Γ_λ of the holomorphic line bundle $\pi : L_\lambda \longrightarrow G_{\mathbb{C}}/B^+$, where B^+ denotes the Borel subgroup of $G_{\mathbb{C}}$ corresponding to the positive roots of $\mathcal{G}_{\mathbb{C}}$. The properties of the space Γ_λ are elucidated by the most famous Borel–Weil theorem for compact groups.

Theorem 2.1 (Borel–Weil theorem of compact groups [1, 3, 9]).

(1) The holomorphic line bundle L_λ has nonzero holomorphic sections if and only if the weight λ of G is the antidominant weight of G.

(2) If λ is the antidominant weight of G, then the holomorphic section space Γ_λ of the holomorphic line bundle L_λ is the irreducible unitary representation space of G with the lowest weight λ.

Remark. The arguments above show that the irreducible unitary representation of G can be obtained through our generalized geometric quantization of the complex homogeneous space G/T of G. Proposition 1.1 and corollary 1.2 imply that the induced Kostant mapping makes the induced representation of G coincide with the geometric quantization of G/T. For more details of the calculation in the case of compact groups, please refer to refs. [6, 12].

2.2. LOOP GROUPS AND THEIR CENTRAL EXTENSIONS

Suppose that G is a connected, simply connected and compact semisimple Lie group. Let LG be the loop group of G, and $\tilde{L}G$ be the central extension of LG determined by the basic two-cocycle ω of LG in ref. [9].

$\tilde{L}G$ can be regarded as the different principal bundles on its different homogeneous spaces. In this paper, we would like to discuss the following two fundamental homogeneous spaces [10]:

1. LG/T: the principal bundle $\mathbf{T} \times T \longrightarrow \tilde{L}G \longrightarrow LG/T$, where $\mathbf{T}$, T denote the center of $\tilde{L}G$, and the maximal torus of G, respectively.

2. LG/G: the principal bundle $\mathbf{T} \times G \longrightarrow \tilde{L}G \longrightarrow \Omega G$, where ΩG is the subgroup of LG whose element γ satisfies that $\gamma(1) = e$ where $1 \in S^1$, and e is the unit element of G. In fact, $\Omega G \cong LG/G$.

2.2.1. The principal bundle $\pi : \tilde{L}G \longrightarrow LG/T$. By the structure of the Lie algebra $\tilde{L}\mathcal{G}$ of $\tilde{L}G$, it follows directly that the principal bundle $\mathbf{T} \times T \longrightarrow \tilde{L}G \longrightarrow LG/T$ satisfies the conditions supposed in proposition 1.1 and corollary 1.2.

Then this principal bundle $\pi : \tilde{L}G \longrightarrow LG/T$ can be equipped with the structures of the connection α, and generalized Kostant mapping $\delta_{LG/T}$.

If λ is a weight of $\tilde{L}G$ [9], then we can induce a complex line bundle $\mathbb{C} \longrightarrow L_\lambda \longrightarrow LG/T$ on LG/T from this principal bundle, and impose the induced connection and induced Kostant mapping on the complex line bundle. Thus we regard L_λ as a generalized prequantization line bundle on LG/T. Moreover, by the general theory of Pressley and Segal [10], the fundamental homogeneous space LG/T can be identified with a homogeneous complex manifold $LG_{\mathbb{C}}/B^+G_{\mathbb{C}}$, where $B^+G_{\mathbb{C}}$ consists of the boundary values of holomorphic maps

$$\gamma : \{z : |z| < 1\} \longrightarrow G_{\mathbb{C}}$$

such that $\gamma(0)$ belongs to the Borel subgroup B^+ of $G_{\mathbb{C}}$ corresponding to the positive roots of $\mathcal{G}_{\mathbb{C}}$.

As in the case of compact groups, we choose the polarization P determined by the Lie subalgebra $N^+\mathcal{G}_{\mathbb{C}}$ of $\tilde{L}\mathcal{G}_{\mathbb{C}}$, which consists of the boundary values of holomorphic maps

$$\xi : \{z : |z| < 1\} \longrightarrow \mathcal{G}_{\mathbb{C}}$$

such that $\xi(0)$ belongs to the nilpotent Lie subalgebra $\mathcal{N}^+$ of $\mathcal{G}_{\mathbb{C}}$ corresponding to the positive roots of $\mathcal{G}_{\mathbb{C}}$. By the complex structure of LG/T, it follows that the polarization space

$$\Gamma_0^P(L_\lambda) \stackrel{\text{def}}{=} \{F \in \Gamma(L_\lambda) \mid \nabla^{(\lambda)}_{X(\pi(\gamma))} F(\gamma) = 0 \text{ for all } \gamma \in \tilde{L}G, \text{ and all tangent vectors } X(\pi(\gamma)) \in P(\pi(\gamma))\}$$

is just the holomorphic section space Γ_λ of the holomorphic line bundle L_λ on the homogeneous complex manifold LG/T ($\cong LG_{\mathbb{C}}/B^+G_{\mathbb{C}}$). Being similar to compact groups, the Borel–Weil theorem for loop groups [10] completely describes the space Γ_λ.

Theorem 2.2 (Borel–Weil theorem for loop groups [10]).

(1) The holomorphic line bundle L_λ on the homogeneous complex manifold LG/T possesses non-vanishing holomorphic sections if and only if the weight λ of $\tilde{L}G$ is antidominant.

(2) If the representation Γ_λ is nonzero, then the representation is:

(i) of positive energy,

(ii) of finite type,

(iii) essentially unitary, and

(iv) irreducible, with lowest weight λ.

Remark. The Borel–Weil theorem of loop groups and induced Kostant mapping also identify the induced representation of $\tilde{L}G$ on Γ_λ with the generalized geometric quantization of its infinite dimensional homogeneous complex manifold $LG/T \cong LG_{\mathbb{C}}/B^+G_{\mathbb{C}}$.

2.2.2. The principal bundle $\pi : \tilde{L}G \longrightarrow LG/G(\cong \Omega G)$. The Wess–Zumino–Witten model with the gauge group G plays an important role in two dimensional conformal field theory [2, 4, 16]. Following from the discussions in refs. [11] and [16], the classical phase space of the WZW model is $\Omega G \times \Omega G$. Since the left copy and the right copy of ΩG in this classical phase space are homeomorphic, it is enough to investigate the geometric quantization of ΩG. Moreover, the geometric quantization of the classical phase space $\Omega G \times \Omega G$ is just the direct sum of the geometric quantizations of the two copies of ΩG [16].

With the above arguments, it is natural to take the principal bundle $\mathrm{T} \times G \longrightarrow \tilde{L}G \longrightarrow \Omega G$ as the starting point of the geometric quantization of ΩG. It follows that this principal bundle satisfies the conditions (1.2), (1.9), and (1.10). By proposition 1.1, we obtain an Ehresmann connection and generalized Kostant mappings.

If $\boldsymbol{\lambda} = (0, \lambda, h)$ is an antidominant weight of $\tilde{L}G$ [10], where the level h (a positive integer) is the character of the center T of $\tilde{L}G$, then λ is an antidominant weight of G. By theorem 2.1 (Borel–Weil theorem for compact groups), there exists one and only one irreducible unitary representation $(V_\lambda, \rho_\lambda)$ with the lowest weight λ, where the representation space V_λ is the holomorphic section space Γ_λ of the induced holomorphic line bundle L_λ on the homogeneous complex manifold G/T. With $\boldsymbol{\lambda}$, and the irreducible representation $(V_\lambda, \rho_\lambda)$ of G, the complex vector bundle $V_\lambda \longrightarrow E_{\boldsymbol{\lambda}}^{(r)} \longrightarrow \Omega G$ can be induced from the principal bundle $\mathrm{T} \times G \longrightarrow \tilde{L}G \longrightarrow \Omega G$. Then we can also impose the structures of the induced connection and induced generalized Kostant mappings on this complex vector bundle. Thus we regard the complex vector bundle $E_{\boldsymbol{\lambda}}^{(r)}$ equipped with the structures stated above as the prequantization vector bundle of ΩG.

In order to obtain the polarization of the prequantization vector bundle of ΩG, we first briefly investigate the structure of the homogeneous space $\Omega G \cong LG/G$. As we know from ref. [10], ΩG can be identified with a homogeneous complex manifold $LG_{\mathbb{C}}/L^+G_{\mathbb{C}}$, where $L^+G_{\mathbb{C}}$ consists of the boundary values of the holomorphic maps

$$\gamma : \{z : |z| < 1\} \longrightarrow G_{\mathbb{C}}.$$

Moreover, ΩG is a Kähler manifold whose Kähler form is just the basic two-cocycle ω of LG [9].

Now we choose the polarization P of ΩG as

$$P(\pi(\gamma)) \stackrel{\text{def}}{=} \{\pi_*(L_\gamma(\xi)) \mid \xi \in L_0^+\mathcal{G}_{\mathbb{C}}\} \tag{2.5}$$

for all $\gamma \in \tilde{L}G$, where the Lie subalgebra $L_0^+\mathcal{G}_{\mathbb{C}}$ of $L\mathcal{G}_{\mathbb{C}}$ consists of the boundary values of holomorphic maps

$$\xi : \{z : |z| < 1\} \longrightarrow \mathcal{G}_{\mathbb{C}}$$

such that $\xi(0) = 0$. The polarization space is defined as

$$\Gamma_0^P(E_\lambda^{(r)}) \stackrel{\text{def}}{=} \{ F \in \Gamma(E_\lambda^{(r)}) \mid \nabla^{(\lambda)}_{X(\pi(\lambda))} F(\gamma) = 0 \text{ for all } \gamma \in \tilde{L}G, \text{ and all tangent vectors } X(\pi(\gamma)) \in P(\pi(\gamma))\}, \tag{2.6}$$

where $\Gamma(E_\lambda^{(r)})$ denotes the smooth section space of the vector bundle $E_\lambda^{(r)}$, which can be identified with the following subspace of $C^\infty(\tilde{L}G, V_\lambda)$:

$$\Gamma(E_\lambda^{(r)}) \cong \{ F \in C^\infty(\tilde{L}G, V_\lambda) \mid F(\gamma t) = \boldsymbol{\lambda}(t)^{-1} F(\gamma),\ F(\gamma g) = \rho_\lambda(g).^{-1} F(\gamma) \text{ for all } \gamma \in \tilde{L}G, \text{ all } t \in \mathbf{T}, \text{ and all } g \in G\}. \tag{2.7}$$

Lemma 2.3 (The explicit form of $\Gamma_0^P(E_\lambda^{(r)})$). *Suppose that $\boldsymbol{\lambda}(= (0, \lambda, h))$ is an antidominant weight of $\tilde{L}G$. Let $(V_\lambda, \rho_\lambda)$ be the irreducible unitary representation of $G_{\mathbb{C}}$ induced by the antidominant weight λ of G in theorem 2.1. We extend ρ_λ as the homomorphism $\rho_{\boldsymbol{\lambda}}$ from $L^+G_{\mathbb{C}} \times \mathbb{C}^\times$ to Hom(V_λ, V_λ), which satisfies*

$$\rho_{\boldsymbol{\lambda}}|_{L_1^+G_{\mathbb{C}}} = 1, \qquad \rho_{\boldsymbol{\lambda}}|_{\mathbb{C}^\times} = h, \qquad \rho_{\boldsymbol{\lambda}}|_{G_{\mathbb{C}}} = \rho_\lambda, \tag{2.8}$$

where $\mathbb{C}^\times$ denotes the complexification of the center $\mathbf{T}$ of $\tilde{L}G$,and the subgroup $L_1^+G_{\mathbb{C}}$ of $LG_{\mathbb{C}}$ consists of the boundary of the holomorphic maps

$$\gamma : \{z : |z| < 1\} \longrightarrow G_{\mathbb{C}}$$

such that $\gamma(0) = e \in G$.

We define $E_{\boldsymbol{\lambda}}$ as the induced holomorphic vector bundle from the complex principal bundle

$$L^+G_{\mathbb{C}} \times \mathbb{C}^\times \longrightarrow \tilde{L}G_{\mathbb{C}} \longrightarrow \Omega G(\cong LG_{\mathbb{C}}/L^+G_{\mathbb{C}})$$

by the extended homomorphism $\rho_{\boldsymbol{\lambda}} : L^+G_{\mathbb{C}} \times \mathbb{C}^\times \longrightarrow$ Hom(V_λ, V_λ).

Then the polarization space $\Gamma_0^P(E_\lambda^{(r)})$ of the prequantization vector bundle $E_\lambda^{(r)}$ is just the holomorphic section space $\Gamma_{\text{hol}}(E_{\boldsymbol{\lambda}})$ of the holomorphic vector bundle $V_{\boldsymbol{\lambda}} \longrightarrow E_{\boldsymbol{\lambda}} \longrightarrow \Omega G$ on ΩG.

Proof.

(1) Since for all $F \in \Gamma_0^P(E_\lambda^{(r)})$, all $\xi \in L_0^+\mathcal{G}_{\mathbb{C}}$, and all $\gamma \in \tilde{L}G$, by (2.6),

$$0 = \nabla^{(\lambda)}_{\pi_*(L_\xi(\gamma))}(F)(\gamma) = L_\xi(F)(\gamma) = \frac{\mathrm{d}}{\mathrm{d}t} F(\gamma \exp t\xi)|_{t=0},$$

it follows that $F(\gamma b) = F(\gamma)$ for all $b \in L_1^+G_{\mathbb{C}}$. This implies that $F \in \Gamma_{\text{hol}}(E_{\boldsymbol{\lambda}})$, i.e., $\Gamma_0^P(E_\lambda^{(r)}) \subseteq \Gamma_{\text{hol}}(E_{\boldsymbol{\lambda}})$.

(2) By the complex structure of ΩG, and the definition of the polarization space $\Gamma_0^P(E_\lambda^{(r)})$ in (2.6), it follows directly that $\Gamma_{\text{hol}}(E_{\boldsymbol{\lambda}}) \subseteq \Gamma_0^P(E_\lambda^{(r)})$.

It follows from (1) and (2) above that the identity $\Gamma_0^P(E_\lambda^{(r)}) = \Gamma_{\text{hol}}(E_{\boldsymbol{\lambda}})$ holds. $\square$

Remark. Lemma 2.3 shows that generalized geometric quantization space of ΩG is the holomorphic section space $\Gamma_{\text{hol}}(E_{\boldsymbol{\lambda}})$ of the holomorphic vector bundle $E_{\boldsymbol{\lambda}}$ of ΩG. The question whether $\Gamma_{\text{hol}}(E_{\boldsymbol{\lambda}})$ is non-zero or not needs to be answered. The positive answer to this question is similar to the Borel–Weil theorem for loop groups (theorem 2.2).

Lemma 2.4 (The model space of G [5, 7, 11]). *Suppose that G is a compact semisimple Lie group. We define the model space $\mathcal{A}$ of G as the homogeneous complex manifold $G_{\mathbb{C}}/N^+$, where N^+ is the maximal nilpotent subgroup of G corresponding to the positive root vectors of the complexified Lie algebra $\mathcal{G}_{\mathbb{C}}$ of G. Then the holomorphic function space* Hol$(\mathcal{A})$ *of $\mathcal{A}$ as the representation space of G can be decomposed into the direct sum of the irreducible representation spaces of G in which each irreducible representation appears once and only once.*

Proof. Let T be the maximal torus of G. Since $TN^+ = N^+T$, T has the right actions on the model space $\mathcal{A}$ of G. It follows that the right actions of T on $\mathcal{A}$ commute with the left actions of G on $\mathcal{A}$. Since T is abelian, $\text{Hol}(\mathcal{A})$ can be decomposed into the direct sum $\bigoplus_\lambda H_\lambda$, where H_λ denotes the weight space corresponding to the character λ of T. In particular, H_λ is invariant under the induced actions of G on $\text{Hol}(\mathcal{A})$.

In fact, H_λ is just the holomorphic section space Γ_λ of the induced holomorphic line bundle L_λ of G/T. By theorem 2.1 (Borel–Weil theorem for compact groups), H_λ is the irreducible representation space of G with λ as the lowest weight. It is obvious that $\Gamma_\lambda \subset \text{Hol}(\mathcal{A})$ for each antidominant weight λ of G. Thus each irreducible representation appears once and only once in the direct sum decomposition of $\text{Hol}(\mathcal{A})$. □

Remark.

(1) Lemma 2.4 states that all irreducible representations of G can be naturally combined into the holomorphic function space $\text{Hol}(\mathcal{A})$ of the model space $\mathcal{A}$. This is just the reason why the homogeneous complex manifold $\mathcal{A}(\stackrel{\text{def}}{=} G_C/N^+)$ is called the model space of G.

(2) This lemma is quite different from the Peter–Weyl theorem. The Peter–Weyl theorem shows that $L^2(G)$ (the square integrable function space of G) as the representation space of G can be decomposed into the direct sum of the irreducible representation spaces of G, and each irreducible representation V_λ appears $(\dim V_\lambda)$ times. The Peter–Weyl theorem for compact groups cannot be generalized to the case of infinite dimensional Lie groups.

Theorem 2.5. *If $\boldsymbol{\lambda}(= (0, \lambda, h))$ is the antidominant weight of $\tilde{L}G$, then there exists a linear isomorphism*

$$\varphi : \Gamma_{\boldsymbol{\lambda}} \longrightarrow \Gamma_{\text{hol}}(E_{\boldsymbol{\lambda}}),$$

where Γ_λ denotes the holomorphic section space of the holomorphic line bundle L_λ on LG/T in theorem 2.2. Moreover, $\Gamma_{\text{hol}}(E_\lambda)$ is the irreducible representation of $\tilde{L}G$ with the lowest weight $\boldsymbol{\lambda}$.

Proof. In the proof of lemma 2.2, the space

$$H_\lambda = \{ f \in \operatorname{Hol}(G_{\mathbb{C}}/N^+) \mid f([gt]) = \lambda(t)^{-1} f([g]) \text{ for all } g \in G_{\mathbb{C}}, \text{ and all } t \in T_{\mathbb{C}} \} \tag{2.9}$$

is just the holomorphic section space Γ_λ of the holomorphic line bundle L_λ on G/T in theorem 2.1. Thus we take H_λ as V_λ in lemma 2.3. It also follows directly that the following identities hold:

$$\Gamma_\lambda \cong \{ f \in \operatorname{Hol}(\tilde{L}G_{\mathbb{C}}/N^+G_{\mathbb{C}}) \mid f([\gamma t]) = \boldsymbol{\lambda}(t)^{-1} f([\gamma]) \text{ for all } \gamma \in \tilde{L}G, \text{ and all } t \in T_{\mathbb{C}} \times \mathbb{C}^\times \}, \tag{2.10}$$

$$\begin{aligned}
\Gamma_{\text{hol}}(E_\lambda) \cong\ & \{ f \in \operatorname{Hol}(\tilde{L}G_{\mathbb{C}}/L_1^+G_{\mathbb{C}}, V_\lambda) \mid f([\gamma g]) = \rho_\lambda(g).^{-1} f([\gamma]) \\
& \text{for all } \gamma \in \tilde{L}G_{\mathbb{C}}, \text{ and all } g \in G_{\mathbb{C}} \times \mathbb{C}^\times \} \\
\cong\ & \{ f \in \operatorname{Hol}(\tilde{L}G_{\mathbb{C}}/L_1^+G_{\mathbb{C}}, \operatorname{Hol}(G_{\mathbb{C}}/N^+)) \mid \\
& f([\gamma g], [g']) = f([\gamma], [gg']), \\
& f([\gamma t], [g]) = \boldsymbol{\lambda}(t)^{-1} f([\gamma], [g]) \\
& \text{for all } \gamma \in \tilde{L}G_{\mathbb{C}}, \text{ all } g, g' \in G_{\mathbb{C}}, \text{ and all } t \in T_{\mathbb{C}} \times \mathbb{C}^\times \}.
\end{aligned} \tag{2.11}$$

We define the linear map

$$\varphi : \Gamma_\lambda \longrightarrow \Gamma_{\text{hol}}(E_\lambda), \quad f \mapsto \varphi(f),$$
$$\varphi(f)([\gamma], [g]) \stackrel{\text{def}}{=} f([\gamma g]) \quad \text{for all } \gamma \in \tilde{L}G_{\mathbb{C}}, \text{ all } g \in G_{\mathbb{C}}. \tag{2.12}$$

By the identities (2.10) and (2.11), it implies that φ is a linear isomorphism. Since $\boldsymbol{\lambda}$ is an antidominant weight of $\tilde{L}G$, it follows from theorem 2.2 that Γ_λ is the irreducible representation of $\tilde{L}G$ with the lowest weight $\boldsymbol{\lambda}$. Thus $\Gamma_{\text{hol}}(E_\lambda)$ is also the irreducible representation of $\tilde{L}G$ with the lowest weight $\boldsymbol{\lambda}$ of $\tilde{L}G$. □

Remark. The induced irreducible representation of $\tilde{L}G$ on $\Gamma_{\text{hol}}(E_\lambda)$ coincides with the generalized geometric quantization of ΩG by theorem 2.5, and induced Kostant mappings.

Theorem 2.6 (The model spaces of $\tilde{L}G$ [11]). *We define two spaces $\mathcal{A}_1$ and $\mathcal{A}_2$ of $\tilde{L}G$ as*

$$\mathcal{A}_1 \stackrel{\text{def}}{=} \tilde{L}G_{\mathbb{C}}/N^+G_{\mathbb{C}}, \qquad \mathcal{A}_2 \stackrel{\text{def}}{=} \tilde{L}G_{\mathbb{C}}/L_1^+G_{\mathbb{C}}, \tag{2.13}$$

respectively. Then:

(1) the holomorphic function space $\operatorname{Hol}(\mathcal{A}_1)$ *of $\mathcal{A}_1$ can be decomposed into the direct sum of the irreducible representations of $\tilde{L}G$ in which each irreducible representation appears once and only once.*

(2) We define a linear space as

$$\mathcal{H}_{\mathcal{A}_2} \stackrel{\text{def}}{=} \{\mathrm{Hol}(\tilde{L}G_{\mathbb{C}}/L_1^+G_{\mathbb{C}},\ \mathrm{Hol}(G_{\mathbb{C}}/N^+)) \mid f([\gamma g],\ [g']) = f([\gamma],\ [gg']) \\ \text{for all } \gamma \in \tilde{L}G_{\mathbb{C}},\ \text{all } g, g' \in G_{\mathbb{C}}\}. \tag{2.14}$$

Then there exists a linear isomorphism $\varphi : \mathrm{Hol}(\mathcal{A}_1) \longrightarrow \mathcal{H}_{\mathcal{A}_2}$, which is defined in (2.12). Moreover, $\mathcal{H}_{\mathcal{A}_2}$ can be decomposed into the direct sum of the irreducible representations of $\tilde{L}G$ in which each irreducible representation appears only once.

Proof.

(1) Similar to the proof of lemma 2.4, it follows directly from theorem 2.2 (Borel–Weil theorem for loop groups) that the statement about $\mathrm{Hol}(\mathcal{A}_1)$ is true.

(2) By lemma 2.4, $\mathrm{Hol}(G_{\mathbb{C}}/N^+) = \bigoplus_\lambda V_\lambda$, where V_λ is the irreducible representation of G with the lowest weight λ of G, and λ runs over the set of antidominant weights of G. Since the following identity

$$f([\gamma g]) = \rho(g).^{-1} f([\gamma]) \tag{2.15}$$

holds for all $f \in \mathcal{H}_{\mathcal{A}_2}$, all $\gamma \in \tilde{L}G_{\mathbb{C}}$, and all $g \in G_{\mathbb{C}}$, where ρ is the representation of $G_{\mathbb{C}}$ on $\mathrm{Hol}(G_{\mathbb{C}}/N^+)$ induced by the left actions of $G_{\mathbb{C}}$ on $G_{\mathbb{C}}/N^+$, the space $\mathcal{H}_{\mathcal{A}_2}$ can be decomposed into the direct sum $\bigoplus_\lambda \mathcal{H}_{\mathcal{A}_2}^{(\lambda)}$, where

$$\mathcal{H}_{\mathcal{A}_2}^{(\lambda)} = \{ f \in \mathrm{Hol}(\tilde{L}G_{\mathbb{C}}/L_1^+G_{\mathbb{C}},\ V_\lambda) \mid f([\gamma g]) = \rho_\lambda(g).^{-1} f([\gamma]) \\ \text{for all } \gamma \in \tilde{L}G_{\mathbb{C}},\ \text{and all } g \in G_{\mathbb{C}}\} \tag{2.16}$$

and λ runs over all antidominant weights of G.

By the fact that the center $\mathbf{T}$ of $\tilde{L}G$ commutes with $L_1^+G_{\mathbb{C}}$, the center $\mathbf{T}$ has right actions on the model space $\mathcal{A}_2 = \tilde{L}G_{\mathbb{C}}/L_1^+G_{\mathbb{C}}$. Thus the space $\mathcal{H}_{\mathcal{A}_2}^{(\lambda)}$ under the right actions of $\mathbf{T}$ can be decomposed into the direct sum $\bigoplus_h \mathcal{H}_{\mathcal{A}_2}^{(\lambda,\ h)}$, where the positive integer h is the character of $\mathbf{T}$, and $\mathcal{H}_{\mathcal{A}_2}^{(\lambda,\ h)}$ is the corresponding characteristic space of $\mathbf{T}$. In fact, the space $\mathcal{H}_{\mathcal{A}_2}^{(\lambda,\ h)}$ can be identified with the holomorphic section space $\Gamma_{\mathrm{hol}}(E_{\boldsymbol{\lambda}})$ of the holomorphic vector bundle $E_{\boldsymbol{\lambda}}$ defined in lemma 2.3 by the weight $\boldsymbol{\lambda}$ of $\tilde{L}G$. By theorem 2.5, $\mathcal{H}_{\mathcal{A}_2}^{(\lambda,\ h)}$ is nonzero if and only if $\boldsymbol{\lambda}$ is an antidominant weight. And on the other hand, if $\boldsymbol{\lambda}(= (0,\ \lambda,\ h))$ is an antidominant weight, then it is easy to see that

$$\Gamma_{\mathrm{hol}}(E_{\boldsymbol{\lambda}}) = \mathcal{H}_{\mathcal{A}_2}^{(\lambda,\ h)} \subset \mathcal{H}_{\mathcal{A}_2}^{(\lambda)} \subset \mathcal{H}_{\mathcal{A}_2}.$$

Thus $\mathcal{H}_{\mathcal{A}_2}$ can be decomposed into the direct sum of the irreducible representations of $\tilde{L}G$ in which each irreducible representation appears only once.

By the definition of φ in (2.15), it follows that φ is a linear isomorphism. Moreover, the statement about $\mathcal{H}_{\mathcal{A}_2}$ in theorem 2.6(2) can also be proved by theorem 2.6(1) through the linear isomorphism φ. □

Remark. Theorem 2.6 shows that the model spaces $\mathcal{A}_1$ and $\mathcal{A}_2$ both can combine the irreducible representations of $\tilde{L}G$ into a global structure such as $\mathrm{Hol}(\mathcal{A}_1)$ and $\mathcal{H}_{\mathcal{A}_2}$.

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